Permutations and Combinations

Learning & Revision for the Day

- Fundamental Principle of Counting
- Factorial Notation
- Permutations
- Circular Permutations
- Combinations
- Applications of Permutations and Combinations
- Prime Factors
- Division of Objects into Groups

Fundamental Principle of Counting

The fundamental principle of counting is a way to figure out the total number of ways in which different events can occur. If a certain work A can be done in m ways and another work B in n ways, then

(i) the number of ways of doing the work C, which is done only when either A or B is done, is $m + n$. (addition principle)

(ii) the number of ways of doing the work C, which is done only when both A and B are done, is mn. (multiplication principle)

Factorial Notation

The product of first n natural numbers is denoted by $n!$ and read as 'factorial n'.

Thus, $n! = n(n-1)(n-2)\dots 3\cdot 2\cdot 1$

Properties of Factorial Notation

1. $0! = 1! = 1$
2. Factorials of negative integers and fractions are not defined.
3. $n! = n(n-1)! = n(n-1)(n-2)!$
4. $\dfrac{n!}{r!} = n(n-1)(n-2)\dots(r+1)$

Permutations

- Permutation means **arrangement** of things. The number of permutations of n different things taken r at a time is ${}^nP_r$.
- ${}^nP_r = n(n-1)(n-2)\dots(n-r+1) = \dfrac{n!}{(n-r)!}, 0 \le r \le n$

Properties of ^nP_r

(i) $^nP_0 = 1, {}^nP_1 = n, {}^nP_n = n!$ (ii) $^nP_r + r \cdot {}^nP_{r-1} = {}^{n+1}P_r$

(iii) $^nP_r = n\, {}^{n-1}P_{r-1}$ (iv) $^nP_r = (n - r + 1)\, {}^nP_{r-1}$

(v) $^{n-1}P_r = (n - r)\, {}^{n-1}P_{r-1}$

Important Results on Permutations

(i) Number of permutations of n different things taken r at a time when a particular thing is to be always included in each arrangement is $r \cdot {}^{n-1}P_{r-1}$.

(ii) Number of permutations of n different things taken r at a time, when a particular thing is never taken in each arrangement is $^{n-1}P_r$.

(iii) The number of permutations of n different things taken r at a time, allowing repetitions is n^r.

(iv) The permutations of n things of which p are identical of one sort, q are identical of second sort, r are identical of third sort, is $\dfrac{n!}{p!\,q!\,r!}$, where $p + q + r = n$.

(v) **Arrangements**

(a) The number of ways in which m different things and n different things $(m + 1 \geq n)$ can be arranged in a row, so that no two things of second kind come together is $m!\, {}^{(m+1)}P_n$.

(b) The number of ways in which m different things and n different things $(m \geq n)$ can be arranged in a row so that all the second type of things come together is $(m + 1)!\, n!$.

(vi) **Dearrangement** The number of dearrangements (No object goes to its scheduled place) of n objects, is

$$n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots (-1)^n \frac{1}{n!}\right].$$

(vii) **Sum of Digits**

(a) Sum of numbers formed by taking all the given n digits (excluding 0) is (sum of all the n digits) $\times (n-1)! \times (111 \ldots n$ times).

(b) Sum of the numbers formed by taking all the given n digits (including 0) is (sum of all the n digits) $\times [(n-1)! \times (111 \ldots n \text{ times}) - (n-2) \times \{111 \ldots (n-1) \text{ times}\}]$.

Circular Permutations

- If different objects are arranged along a closed curve, then permutation is known as circular permutation.
- The number of circular permutations of n different things taken all at a time is $(n-1)!$. If clockwise and anti-clockwise orders are taken as different.
- If clockwise and anti-clockwise circular permutations are considered to be same, then it is $\dfrac{(n-1)!}{2}$.
- Number of circular permutations of n different things, taken r at a time, when clockwise and anti-clockwise orders are taken as different, is $\dfrac{^nP_r}{r}$.
- Number of circular permutations of n different things, taken r at a time, when clockwise and anti-clockwise orders are not different, is $\dfrac{^nP_r}{2r}$.

Important Results on Circular Permutations

(i) The number of ways in which m different things and n different things (where, $m \geq n$) can be arranged in a circle, so that no two things of second kind come together is $(m-1)!\, {}^mP_n$.

(ii) The number of ways in which m different things and n different things can be arranged in a circle, so that all the second type of things come together is $m!\, n!$.

(iii) The number of ways in which m different things and n different things (where, $m \geq n$) can be arranged in the form of garland, so that no two things of second kind come together is $(m-1)!\, {}^mP_n/2$.

(iv) The number of ways in which m different things and n different things can be arranged in the form of garland, so that all the second type of things come together is $m!\, n!/2$.

Combinations

Combination means **selection** of things. The number of combinations of n different things taken r at a time is ^nC_r or $\dbinom{n}{r}$.

$$^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}$$

Properties of ^nC_r

(i) $^nC_r = {}^nC_{n-r}$ (ii) $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Important Results on Combinations

(i) The number of combinations of n different things, taken r at a time, when p particular things always occur is $^{n-p}C_{r-p}$.

(ii) The number of combinations of n different things, taken r at a time, when p particular things never occur is $^{n-p}C_r$.

(iii) The number of selections of zero or more things out of n different things is $^nC_0 + {}^nC_1 + \ldots + {}^nC_n = 2^n$.

(iv) The number of selections of one or more things out of n different things is $^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1$.

(v) The number of selections of zero or more things out of n identical things $= n + 1$.

(vi) The number of selections of one or more things out of n identical things $= n$.

(vii) The number of selections of one or more things from $p + q + r$ things, where p are alike of one kind, q are alike of second kind and rest are alike of third kind, is $[(p+1)(q+1)(r+1)] - 1$.

(viii) The number of selections of one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things, is $(p+1)(q+1)(r+1)2^n - 1$

(ix) If there are m items of one kind, n items of another kind and so on. Then, the number of ways of choosing r items out of these items = coefficient of x^r in
$$(1 + x + x^2 + \dots + x^m)(1 + x + x^2 + \dots + x^n)\dots$$

(x) If there are m items of one kind, n items of another kind and so on. Then, the number of ways of choosing r items out of these items such that atleast one item of each kind is included in every selection = coefficient of x^r in
$$(x + x^2 + \dots + x^m)(x + x^2 + \dots + x^n)\dots$$

Applications of Permutations and Combinations

Functional Applications

If the set A has m elements and B has n elements, then

(i) the number of functions from A to B is n^m.

(ii) the number of one-one functions from A to B is ${}^nP_m$, $m \le n$.

(iii) the number of onto functions from A to B is
$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \dots, m \le n.$$

(iv) the number of bijections from A to B is $n!$, if $m = n$.

Geometrical Applications

(i) Number of triangles formed from n points, when no three points are collinear, is ${}^nC_3$.

(ii) Out of n non-concurrent and non-parallel straight lines, the points of intersection are ${}^nC_2$.

(iii) The number of diagonals in a polygon of n sides is ${}^nC_2 - n$.

(iv) The number of total triangles formed by the n points on a plane of which m are collinear, is ${}^nC_3 - {}^mC_3$.

(v) The number of total different straight lines, formed by the n points on a plane, of which m are collinear, is ${}^nC_2 - {}^mC_2 + 1$.

Prime Factors

Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where $p_i, i = 1, 2, \dots, r$ are distinct primes and α_i, $i = 1, 2, \dots, r$ are positive integers.

(i) Number of divisor of n is
$$(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\dots(\alpha_r + 1).$$

(ii) Sum of divisors of n is $\dfrac{(p_1^{\alpha_1+1} - 1)}{(p_1 - 1)} \dfrac{(p_2^{\alpha_2+1} - 1)}{(p_2 - 1)} \dots \dfrac{(p_r^{\alpha_r+1} - 1)}{(p_r - 1)}$

(iii) If p is a prime such that p^r divides $n!$ but p^{r+1} does not divide $n!$.

Then, $r = \left[\dfrac{n}{p}\right] + \left[\dfrac{n}{p^2}\right] + \dots$

Division of Objects into Groups

Objects are Different

(i) The number of ways of dividing n different objects into 3 groups of size p, q and r $(p + q + r = n)$ is

(a) $\dfrac{n!}{p!\,q!\,r!}$; p, q and r are unequal.

(b) $\dfrac{n!}{p!\,2!\,(q!)^2}$; $q = r$

(c) $\dfrac{n!}{3!\,(p!)^3}$; $p = q = r$

(ii) The number of ways in which n different things can be distributed into r different groups, if empty groups are allowed, is r^n.

(iii) The number of ways in which n different things can be distributed into r different groups, if empty groups are not allowed, is
$$r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \dots + (-1)^{r-1}\,{}^rC_{r-1} \cdot 1$$

Objects are Identical

(i) The number of ways of dividing n identical objects among r persons such that each one may get atmost n objects, is $\binom{n+r-1}{r-1}$. In other words, the total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

(ii) The total number of ways of dividing n identical objects among r persons, each one of whom, receives atleast one item, is ${}^{n-1}C_{r-1}$. In other words, the number of ways in which n identical things can be divided into r groups such that blank groups are not allowed, is ${}^{n-1}C_{r-1}$.

(iii) Number of non-negative integral solutions of the equation $x_1 + x_2 + \dots + x_r = n$ is equivalent to number of ways of distributing n identical objects into r groups if empty groups are allowed, which is ${}^{n+r-1}C_{r-1}$.

(iv) Number of positive integral solutions of the equation $x_1 + x_2 + \dots + x_r = n$ is equivalent to the number of ways of distributing n identical objects into r groups such that no group empty, which is ${}^{n-1}C_{r-1}$.

(v) Number of integral solutions of the equation $x_1 + x_2 + \dots + x_r = n$, where $a \le x_i \le b, \forall\, i = 1, 2, \dots, r$, is given by coefficient of x^n in $(x^a + x^{a+1} + x^{a+2} + \dots + x^b)^r$.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 The value of $2^n [1 \cdot 3 \cdot 5 \ldots (2n-3)(2n-1)]$ is

(a) $\frac{(2n)!}{n!}$ (b) $\frac{(2n)!}{2^n}$ (c) $\frac{n!}{(2n)!}$ (d) None of these

2 The value of $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n!$, is

(a) $(n+1)!$ (b) $(n+1)!+1$
(c) $(n+1)!-1$ (d) None of these

3 How many different nine-digit numbers can be formed from the digits of the number 223355888 by rearrangement of the digits so that the odd digits occupy even places?

(a) 16 (b) 36 (c) 60 (d) 180

4 A library has a copies of one book, b copies of each of two books, c copies of each of three books and single copies of d books. The total number of ways in which these books can be arranged, is

(a) $\frac{(a+b+c+d)!}{a!\,b!\,c!}$ (b) $\frac{(a+2b+3c+d)!}{a!\,(b!)^2\,(c!)^3}$
(c) $\frac{(a+2b+3c+d)}{a!\,b!\,c!}$ (d) None of these

5 The number of words which can be formed out of the letters of the word ARTICLE, so that vowels occupy the even place is ➞ **NCERT Exemplar**

(a) 1440 (b) 144 (c) 7! (d) ${}^4C_4 \times {}^3C_3$

6 In how many ways the letters of the word 'ARRANGE' can be arranged without altering the relative positions of vowels and consonants?

(a) 36 (b) 26
(c) 62 (d) None of these

7 If the letters of the word 'SACHIN' are arranged in all possible ways and these words are written in dictionary order, then the word 'SACHIN' appears at serial number

(a) 600 (b) 601 (c) 602 (d) 603

8 The number of integers greater than 6000 that can be formed, using the digits 3, 5, 6, 7 and 8 without repetition, is

(a) 216 (b) 192 (c) 120 (d) 72

9 The number of 5-digits telephone numbers having atleast one of their digits repeated, is ➞ **NCERT Exemplar**

(a) 90000 (b) 100000 (c) 30240 (d) 69760

10 If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in dictionary, then the position of the word SMALL is

(a) 46th (b) 59th
(c) 52nd (d) 58th

11 The total number of permutations of $n\,(>1)$ different things taken not more than r at a time, when each thing may be repeated any number of times is

(a) $\frac{n(n^n-1)}{n-1}$ (b) $\frac{n^r-1}{n-1}$
(c) $\frac{n(n^r-1)}{n-1}$ (d) None of these

12 If $\frac{{}^nP_{r-1}}{a} = \frac{{}^nP_r}{b} = \frac{{}^nP_{r+1}}{c}$, then

(a) $b^2 = a(b+c)$ (b) $c^2 = a(b+c)$
(c) $ab = a^2 + bc$ (d) $bc = a^3 + b^2$

13 The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

(a) {1, 2, 3} (b) {1, 2, 3, 4, 5, 6}
(c) {1, 2, 3, 4} (d) {1, 2, 3, 4, 5}

14 Find the number of different words that can be formed from the letters of the word TRIANGLE, so that no vowels are together.

(a) 14000 (b) 14500 (c) 14400 (d) 14402

15 In a class of 10 students, there are 3 girls. The number of ways they can be arranged in a row, so that no 2 girls are consecutive is $k \cdot 8!$, where k is equal to

(a) 12 (b) 24 (c) 36 (d) 42

16 The sum of all the 4-digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 without repetition of the digits is

(a) 399960 (b) 288860 (c) 301250 (d) 420210

17 If eleven members of a committee sit at a round table so that the President and Secretary always sit together, then the number of arrangements is

(a) $10! \times 2$ (b) $10!$
(c) $9! \times 2$ (d) None of these

18 Let $f: \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ that are onto and $f(x) \neq x$, is equal to

(a) 9 (b) 44
(c) 16 (d) None of these

19 There are 4 balls of different colours and 4 boxes of same colours as those of the balls. The number of ways in which the balls, one in each box, could be placed such that a ball does not go to box of its own colour is

(a) 8 (b) 9 (c) 7 (d) 1

20 How many different words can be formed by jumbling the letters in the word 'MISSISSIPPI' in which no two S are adjacent?

(a) $7 \cdot {}^6C_4 \cdot {}^8C_4$ (b) $8 \cdot {}^6C_4 \cdot {}^7C_4$
(c) $6 \cdot 7 \cdot {}^8C_4$ (d) $6 \cdot 8 \cdot {}^7C_4$

21 The number of ways in which seven persons can be arranged at a round table, if two particular persons may not sit together is

(a) 480 (b) 120
(c) 80 (d) None of these

22 The number of ways in which 6 men and 5 women can sit at a round table, if no two women are to sit together, is given by

(a) $6! \times 5!$ (b) 30 (c) $5! \times 4!$ (d) $7! \times 5!$

23 The value of $^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$ is equal to

(a) $^{47}C_6$ (b) $^{52}C_5$
(c) $^{52}C_4$ (d) None of these

24 If $^nC_3 + {}^nC_4 > {}^{n+1}C_3$, then

(a) $n > 6$ (b) $n > 7$
(c) $n < 6$ (d) None of these

25 The number of ways in which we can choose a committee from four men and six women so that the committee includes atleast two men and exactly twice as many women as men is ➡ **NCERT Exemplar**

(a) 94 (b) 126
(c) 128 (d) None of these

26 A box contains 2 white balls, 3 black balls and 4 red balls. The number of ways of drawing 3 balls from the box, if atleast one black ball is included, is

(a) 36 (b) 42 (c) 56 (d) 64

27 A student is allowed to select atmost n books from a collection of $(2n+1)$ books. If the number of ways in which he can select atleast one book is 63, then n is equal to

(a) 3 (b) 4 (c) 6 (d) 5

28 Let A and B two sets containing 2 elements and 4 elements, respectively. The number of subsets of $A \times B$ having 3 or more elements is

(a) 256 (b) 220 (c) 219 (d) 211

29 In an examination of 9 papers a candidate has to pass in more papers than the number of papers in which he fails, in order to be successful. The number of ways in which he can be unsuccessful is

(a) 255 (b) 256 (c) 193 (d) 319

30 There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn, two balls are taken out at random and then transferred to the other. The number of ways in which this can be done, is

(a) 3 (b) 36 (c) 66 (d) 108

31 If the total number of m elements subsets of the set $A = \{a_1, a_2, a_3, \ldots, a_n\}$ is λ times the number of m elements subsets containing a_4, then n is

(a) $(m-1)\lambda$ (b) $m\lambda$ (c) $(m+1)\lambda$ (d) 0

32 A guard of 12 men is formed from a group of n soldiers in all possible ways. If the number of times two particular soldiers A and B are together on guard is thrice the number of times three particular soldiers C, D, E are together on guard, then n is equal to

(a) 18 (b) 24 (c) 32 (d) 36

33 In a steamer, there are stalls for 12 animals and there are horses, cows and calves (not less than 12 each) ready to be shipped. In how many ways, can the ship load be made?

(a) $3^{12} - 1$ (b) 3^{12} (c) $(12)^3 - 1$ (d) $(12)^3$

34 There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then

(a) $N > 190$ (b) $N \le 100$
(c) $100 < N \le 140$ (d) $140 < N \le 190$

35 Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is

(a) 7 (b) 5
(c) 10 (d) 8

36 On the sides AB, BC, CA of a ΔABC, 3, 4, 5 distinct points (excluding vertices A, B, C) are respectively chosen. The number of triangles that can be constructed using these chosen points as vertices are

(a) 210 (b) 205
(c) 215 (d) 220

37 The number of divisors of the number of 38808 (excluding 1 and the number itself) is

(a) 70 (b) 72
(c) 71 (d) None of these

38 If a, b, c, d, e are prime integers, then the number of divisors of ab^2c^2de excluding 1 as a factor is

(a) 94 (b) 72 (c) 36 (d) 71

39 The number of ways of distributing 8 identical balls in 3 distinct boxes, so that no box is empty, is

(a) 5 (b) $\binom{8}{3}$
(c) 3^8 (d) 21

40 If 4 dice are rolled, then the number of ways of getting the sum 10 is

(a) 56 (b) 64
(c) 72 (d) 80

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf, so that the dictionary is always in the middle. The number of such arrangements is

(a) atleast 1000 (b) less than 500
(c) atleast 500 but less than 750
(d) atleast 750 but less than 1000

2 Sixteen men compete with one another in running, swimming and riding. How many prize lists could be made, if there were altogether 6 prizes for different values, one for running, 2 for swimming and 3 for riding?

(a) $16 \times 15 \times 14$ (b) $16^3 \times 15^2 \times 14$
(c) $16^3 \times 15 \times 14^2$ (d) $16^2 \times 15 \times 14$

3 The number of ways of dividing 52 cards amongst four players so that three players have 17 cards each and the fourth players just one card, is

(a) $\frac{52!}{(17!)^3}$ (b) 52! (c) $\frac{52!}{17!}$ (d) None of these

4 If the letters of the word MOTHER are written in all possible orders and these words are written out as in a dictionary, then the rank of the word MOTHER is

(a) 240 (b) 261 (c) 308 (d) 309

5 Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22} \right\}$. Then, the number of non-singular matrices in the set S, is

(a) 27 (b) 24 (c) 10 (d) 20

6 A group of 6 is chosen from 10 men and 7 women so as to contain atleast 3 men and 2 women. The number of ways this can be done, if two particular women refuse to serve on the same group, is

(a) 8000 (b) 7800 (c) 7600 (d) 7200

7 A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then, the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is

(a) 485 (b) 468 (c) 469 (d) 484

8 The number of ways in which we can select four numbers from 1 to 30 so as to exclude every selection of four consecutive numbers is

(a) 27378 (b) 27405 (c) 27399 (d) None of these

9 The number of numbers divisible by 3 that can be formed by four different even digits is

(a) 36 (b) 18 (c) 0 (d) None of these

10 The total number of integral solutions (x, y, z) such that $xyz = 24$ is

(a) 36 (b) 90 (c) 120 (d) None of these

11 Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is

(a) 880 (b) 629 (c) 630 (d) 879

12 The number of ways in which an examiner can assign 30 marks to 8 questions, giving not less than 2 marks to any question, is

(a) ${}^{30}C_7$ (b) ${}^{21}C_8$ (c) ${}^{21}C_7$ (d) ${}^{30}C_8$

13 The number of divisors of the form $4n + 1, n \geq 0$ of the number $10^{10}\, 11^{11}\, 13^{13}$ is

(a) 750 (b) 840 (c) 924 (d) 1024

14 Out of 8 sailors on a boat, 3 can work only one particular side and 2 only the other side. Then, number of ways in which the sailors can be arranged on the boat is

(a) 2718 (b) 1728 (c) 7218 (d) None of these

15 In a cricket match between two teams X and Y, the team X requires 10 runs to win in the last 3 balls. If the possible runs that can be made from a ball be 0, 1, 2, 3, 4, 5 and 6. The number of sequence of runs made by the batsman is

(a) 12 (b) 18 (c) 21 (d) 36

ANSWERS

SESSION 1	**1.** (a)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (b)	**8.** (b)	**9.** (d)	**10.** (d)
	11. (c)	**12.** (a)	**13.** (a)	**14.** (c)	**15.** (d)	**16.** (a)	**17.** (c)	**18.** (b)	**19.** (b)	**20.** (a)
	21. (a)	**22.** (a)	**23.** (c)	**24.** (a)	**25.** (a)	**26.** (d)	**27.** (a)	**28.** (c)	**29.** (b)	**30.** (d)
	31. (b)	**32.** (c)	**33.** (b)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (a)	**38.** (d)	**39.** (d)	**40.** (d)
SESSION 2	**1.** (a)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (d)	**6.** (b)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (c)
	11. (d)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (d)					

Hints and Explanations

SESSION 1

1 Clearly, $[1\cdot 3\cdot 5 \ldots (2n-3)(2n-1)]2^n$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \ldots (2n-1)(2n)2^n}{2\cdot 4\cdot 6 \ldots 2n}$$

$$= \frac{(2n)!\,2^n}{2^n(1\cdot 2\cdot 3 \ldots n)} = \frac{(2n)!}{n!}$$

2 Clearly, given expression $= \sum_{r=1}^{n} r\cdot r!$

$$= \sum_{r=1}^{n} ((r+1)-1)\,r! = \sum_{r=1}^{n} ((r+1)! - r!)$$

$= (2! - 1!) + (3! - 2!) + \ldots + ((n+1)! - n!)$

$= (n+1)! - 1$

3 In a nine digits number, there are four even places for the four odd digits 3, 3, 5, 5.

$\therefore$ Required number of ways $= \frac{4!}{2!\,2!}\cdot\frac{5!}{2!\,3!}$

$= 60$

4 Total number of books $= a + 2b + 3c + d$

$\therefore$ The total number of arrangements

$$= \frac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$$

5 In a word ARTICLE, vowels are A, E, I and consonants are C, L, R, T.

In a seven letter word, there are three even places in which three vowels are placed in 3! ways. In rest of the four places, four consonants are placed in 4! ways.

$\therefore$ Required number of ways

$= 3! \times 4! = 6 \times 24 = 144$

6 Clearly, the consonants in their positions can be arranged in $\frac{4!}{2!} = 12$ ways and the vowels in their positions can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ Total number of arrangements

$= 12 \times 3 = 36$

7 The letters of given word are A, C, H, I, N, S.

Now, the number of words starting with $A = 5!$

the number of words starting with $C = 5!$

the number of words starting with $H = 5!$

the number of words starting with $I = 5!$

and the number of words starting with $N = 5!$.

Then, next word is SACHIN.

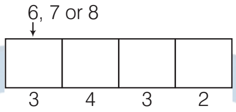
So, the required serial number is

$= (5\cdot 5!) + 1 = 601.$

8 The integer greater than 6000 may be of 4 digits or 5 digits. So, here two cases arise.

***Case* I** When number is of 4 digit.

Four digit number can start from 6,7 or 8.



Thus, total number of 4-digit number, which are greater than

$6000 = 3 \times 4 \times 3 \times 2 = 72$

***Case* II** When number is of 5 digit.

Total number of 5-digit number, which are greater than $6000 = 5! = 120$

$\therefore$ Total number of integers

$= 72 + 120 = 192$

9 Using the digits 0, 1, 2, ..., 9 the number of five digits telephone numbers which can be formed is 10^5 (since, repetition is allowed). The number of five digits telephone numbers, which have none of the digits repeated $= {}^{10}P_5 = 30240$

$\therefore$ The required number of telephone number $= 10^5 - 30240 = 69760$

10 Clearly, number of words start with

$A = \frac{4!}{2!} = 12$

Number of words start with $L = 4! = 24$

Number of words start with $M = \frac{4!}{2!} = 12$

Number of words start with $SA = \frac{3!}{2!} = 3$

Number of words start with $SL = 3! = 6$

Note that, next word will be "SMALL".

Hence, position of word "SMALL" is 58th.

11 When we arrange things one at a time, the number of possible permutations is n. When we arrange them two at a time the number of possible permutations are $n \times n = n^2$ and so on. Thus, the total number of permutations are

$$n + n^2 + \ldots + n^r = \frac{n(n^r - 1)}{n-1} \quad [\because n > 1]$$

12 $\because \frac{{}^nP_{r-1}}{a} = \frac{{}^nP_r}{b} = \frac{{}^nP_{r+1}}{c}$

From first two terms $\frac{b}{a} = n - r + 1$

From last two terms $\frac{c}{b} = n - r$

Hence, $\frac{b}{a} = \frac{c}{b} + 1 \Rightarrow b^2 = a(b+c)$

13 Given that, $f(x) = {}^{7-x}P_{x-3}$. The above function is defined, if $7 - x \geq 0$ and $x - 3 \geq 0$ and $7 - x \geq x - 3$.

$\Rightarrow \quad x \leq 7,\ x \geq 3$ and $x \leq 5$

$\therefore \quad D_f = \{3, 4, 5\}$

Now, $\quad f(3) = {}^4P_0 = 1$

$f(4) = {}^3P_1 = 3$ and $f(5) = {}^2P_2 = 2$

$\therefore \quad R_f = \{1, 2, 3\}$

14 In a word TRIANGLE, vowels are (A, E, I) and consonants are (G, L, N, R, T).

First, we fix the 5 consonants in alternate position in 5! ways.

_G_L_N_R_T_

In rest of the six blank position, three vowels can be arranged in ${}^6P_3$ ways.

$\therefore$ Total number of different words

$= 5! \times {}^6P_3 = 120 \times 6 \times 5 \times 4 = 14400$

15 The 7 boys can be arranged in row in 7! ways. There will be 6 gaps between them and one place before them and one place after them. The 3 girls can be arranged in a row in ${}^8P_3 = 8\cdot 7\cdot 6$ ways

$\therefore$ Required number of ways

$= 7! \times 8\cdot 7\cdot 6 = 42\cdot 8!$

16 Required sum

$=$ (Sum of all the n-digits)

$\times\ {}^{n-1}P_{r-1} \times (111 \ldots r \text{ times})$

$= (1 + 2 + 3 + 4 + 5)\,{}^4P_3 \times (1111)$

$= 15 \times 24 \times 1111 = 399960$

17 Since, out of eleven members two members sit together, the number of arrangements $= 9! \times 2$

($\because$ two members can be sit in two ways.)

18 Total number of functions

$=$ Number of dearrangement of 5 objects

$$= 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

19 Use the number of dearrangements

i.e. $n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right\}$

Here, $n = 4$

So, the required number of ways

$$= 4!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\} = 12 - 4 + 1 = 9$$

20 Given word is MISSISSIPPI.

Here, I = 4 times, S = 4 times,

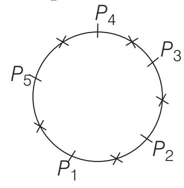
P = 2 times, M = 1 time

_M_I_I_I_I_P_P_

$\therefore$ Required number of words

$$= {}^8C_4 \times \frac{7!}{4!2!} = {}^8C_4 \times \frac{7\times 6!}{4!2!} = 7\cdot {}^8C_4\cdot {}^6C_4$$

21 Clearly, remaining 5 persons can be seated in 4! ways. Now, on five cross marked places person can sit in ${}^5P_2$ ways.

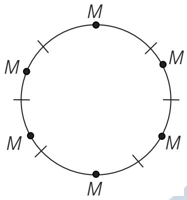


So, number of arrangements

$= 4! \times \frac{5!}{3!} = 24 \times 20 = 480$ ways

22 First, we fix the position of men, number of ways in which men can sit = 5!.

Now, the number of ways in which women can sit = ${}^6P_5$



$\therefore$ Total number of ways

$= 5! \times {}^6P_5 = 5! \times 6!$

23 ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3 = {}^{47}C_4 + {}^{51}C_3$

$+ {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + ({}^{47}C_3 + {}^{47}C_4)$

$= {}^{52}C_4$

24 ${}^nC_3 + {}^nC_4 > {}^{n+1}C_3$

$\Rightarrow {}^{n+1}C_4 > {}^{n+1}C_3 \ ({}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$\Rightarrow \frac{{}^{n+1}C_4}{{}^{n+1}C_3} > 1 \Rightarrow \frac{n-2}{4} > 1 \Rightarrow n > 6$

25 The number of ways in which we can choose a committee

= Choose two men and four women

+ Choose three men and six women

$= {}^4C_2 \times {}^6C_4 + {}^4C_3 \times {}^6C_6$

$= 6 \times 15 + 4 \times 1 = 90 + 4 = 94$

26 The number of ways of drawing 1 black and 2 non-black balls is

${}^3C_1 \cdot {}^6C_2 = 3 \cdot 15 = 45$

The number of ways of drawing 2 black and 1 non-black ball is

${}^3C_2 \cdot {}^6C_1 = 3 \cdot 6 = 18$

The number of ways of drawing 3 black balls is ${}^3C_3 = 1$

$\therefore$ Number of ways $= 45 + 18 + 1 = 64$

27 He can select 1, 2, ... or n books.

The number of ways to select atleast one book is

${}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n$

$= \frac{1}{2}({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n$

$+ {}^{2n+1}C_{n+1} + \ldots + {}^{2n+1}C_{2n})$

$= \frac{1}{2}(2^{2n+1} - {}^{2n+1}C_0 - {}^{2n+1}C_{2n+1})$

$= 2^{2n} - 1 = 63$ [given]

$\Rightarrow 2^{2n} = 64 = 2^6 \Rightarrow n = 3$

28 Given, $n(A) = 2$, $n(B) = 4$.

$\therefore n(A \times B) = 8$

The number of subsets of $A \times B$ having 3 or more elements $= {}^8C_3 + {}^8C_4 + \ldots + {}^8C_8$

$= ({}^8C_0 + {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8)$

$- ({}^8C_0 + {}^8C_1 + {}^8C_2)$

$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2$

$= 256 - 1 - 8 - 28 = 219$

$[\because 2^n = {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n]$

29 Clearly, the candidate is unsuccessful, if he fails in 9 or 8 or 7 or 6 or 5 papers.

$\therefore$ Numbers of ways to be unsuccessful

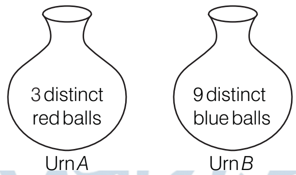
$= {}^9C_9 + {}^9C_8 + {}^9C_7 + {}^9C_6 + {}^9C_5$

$= {}^9C_0 + {}^9C_1 + {}^9C_2 + {}^9C_3 + {}^9C_4$

$= \frac{1}{2}({}^9C_0 + {}^9C_1 + \ldots + {}^9C_9)$

$= \frac{1}{2}(2^9) = 2^8 = 256$

30



The number of ways in which 2 balls from urn A and 2 balls from urn B can be selected $= {}^3C_2 \times {}^9C_2 = 3 \times 36 = 108$

31 Total number of m elements subsets of

$A = {}^nC_m$ ...(i)

and number of m elements subsets of A each containing element $a_4 = {}^{n-1}C_{m-1}$

According to the question,

$= \lambda \cdot {}^{n-1}C_{m-1}$

$\Rightarrow \frac{n}{m} \cdot {}^{n-1}C_{m-1} = \lambda \cdot {}^{n-1}C_{m-1}$

$\Rightarrow \lambda = \frac{n}{m} \Rightarrow n = m\lambda$

32 Number of times A and B are together on guard is $\binom{n-2}{10}$.

Number of times C, D and E are together on guard is $\binom{n-3}{9}$.

According to the question,

$\binom{n-2}{10} = 3\binom{n-3}{9}$

$\Rightarrow n - 2 = 30 \Rightarrow n = 32$

33 First stall can be filled in 3 ways, second stall can be filled in 3 ways and so on.

$\therefore$ Number of ways of loading steamer

$= {}^3C_1 \times {}^3C_1 \times \ldots \times {}^3C_1$ (12 times)

$= 3 \times 3 \times \ldots \times 3$ (12 times) $= 3^{12}$

34 If out of n points, m are collinear, then

Number of triangles $= {}^nC_3 - {}^mC_3$

$\therefore$ Required number of triangles

$= {}^{10}C_3 - {}^6C_3 = 120 - 20$

$\Rightarrow N = 100$

35 $T_n = {}^nC_3$, hence $T_{n+1} = {}^{n+1}C_3$

Now, $T_{n+1} - T_n = 10$

$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 10$ [given]

$\Rightarrow \frac{(n+1)\,n\,(n-1)}{3!} - \frac{n\,(n-1)(n-2)}{3!} = 10$

$\Rightarrow \frac{n\,(n-1)}{3!}(n + 1 - n + 2) = 10$

$\Rightarrow \frac{n\,(n-1)}{3!} \times 3 = 10$

$\Rightarrow n^2 - n - 20 = 0 \Rightarrow n = 5$

36 Required number of triangles that can be constructed using these chosen points as vertices $= {}^{12}C_3 - {}^3C_3 - {}^4C_3 - {}^5C_3$

Here, we subtract those cases in which points are collinear

$= 220 - 1 - 4 - 10 = 220 - 15 = 205$

37 Since, $38808 = 2^3 \times 3^2 \times 7^2 \times 11^1$

$\therefore$ Number of divisors

$= 4 \times 3 \times 3 \times 2 - 2 = 72 - 2 = 70$

38 The number of divisors of ab^2c^2de

$= (1 + 1)(2 + 1)(2 + 1)(1 + 1)(1 + 1) - 1$

$= 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2 - 1 = 71$

39 Required number of ways is equal to the number of positive integer solutions of the equation $x + y + z = 8$ which equal to $\binom{8-1}{3-1} = \binom{7}{2} = 21$

40 Coefficient of x^{10} in $(x + x^2 + \ldots + x^6)^4$

= Coefficient of x^6 in $(1 + x + \ldots + x^5)^4$

$= (1 - x^6)^4 (1 - x)^{-4}$ in $(1 - 4x^6 + \ldots)$

$\left[1 + \binom{4}{1}x + \ldots\right]$

Hence, coefficient of x^6 is $\binom{9}{6} - 4 = 80$.

SESSION 2

1 Given, 6 different novels and 3 different dictionaries.

Number of ways of selecting 4 novels from 6 novels is ${}^6C_4 = \frac{6!}{2!4!} = 15$

Number of ways of selecting 1 dictionary from 3 dictionaries is

${}^3C_1 = \frac{3!}{1!2!} = 3$

Number of arrangement of 4 novels and 1 dictionary where dictionary is always in the middle, is 4!

Required number of arrangement

$15 \times 3 \times 4! = 45 \times 24 = 1080$

2 Number of ways of giving one prize for running = 16

Number of ways of giving two prizes for swimming = 16 × 15

Number of ways of giving three prizes for riding = 16 × 15 × 14

$\therefore$ Required ways of giving prizes

$= 16 \times 16 \times 15 \times 16 \times 15 \times 14$

$= 16^3 \times 15^2 \times 14$

3 For the first player, cards can be distributed in the ${}^{52}C_{17}$ ways. Now, out of 35 cards left 17 cards can be distributed for second player in ${}^{52}C_{17}$ ways.

Similarly, for third player in $^{18}C_{17}$ ways. One card for the last player can be distributed in $^{1}C_{1}$ way.
Therefore, the required number of ways for the proper distribution.

$= {}^{52}C_{17} \times {}^{35}C_{17} \times {}^{18}C_{17} \times {}^{1}C_{1}$

$= \dfrac{52!}{35!\,17!} \times \dfrac{35!}{18!17!} \times \dfrac{18!}{17!1!} \times 1! = \dfrac{52!}{(17!)^3}$

4 The number of words starting from E $= 5! = 120$
The number of words starting from H $= 5! = 120$
The number of words starting from ME $= 4! = 24$
The number of words starting from MH $= 4! = 24$
The number of words starting from MOE $= 3! = 6$
The number of words starting from MOH $= 3! = 6$
The number of words starting from MOR $= 3! = 6$
The number of words starting from MOTE $= 2! = 2$
The number of words starting from MOTHER $= 1! = 1$
Hence, rank of the word MOTHER

$= 2(120) + 2(24) + 3(6) + 2 + 1$
$= 309$

5 A matrix whose determinant is non-zero is called a non-singular matrix.
Here, we have

$$S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22} \right\}$$

Clearly, $n(s) = 27$

[$\because$ for $a_{11} = a_{22}$, we have 3 choices, for a_{12}, we have 3 choices and for a_{21}, we have 3 choices]

Now, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$

$\Rightarrow a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = 0$
$\Rightarrow (a_{11})^2 = a_{12}\, a_{21} = 0$ [$\because a_{11} = a_{22}$]
$\Rightarrow a_{12}a_{21} = 0, 1, 4$ [$\because a_{11} \in \{0, 1, 2\}$]

Consider $a_{12}\, a_{21} = 0$, this is possible in 5 cases
$a_{12}a_{21} = 1$, this is possible in only 1 case
$a_{12}\, a_{21} = 4$, this is possible in only 1 case
Thus, number of singular matrices in S are 7.
Hence, number of non-singular matrices in S are $27 - 7 = 20$.

6 Let the men be $M_1, M_2, \ldots, M_{10}$ and women be $W_1, W_2, \ldots, W_7$.
Let W_1 and W_2 do not want to be on the same group. The six members group can contain 4 men and 2 women or 3 men and 3 women.
The number of ways of forming $4M, 2W$ group is

$^{10}C_4\,({}^5C_2 + {}^2C_1 \cdot {}^5C_1) = 4200$

where, $^5C_2$ is the number of ways without W_1 and W_2 and $^5C_1$ is the number of ways with W_1 and without W_2 or with W_2 and without W_1.
The number of ways of forming $3M, 3W$ group is $^{10}C_3\,({}^5C_3 + {}^2C_1\,{}^5C_2) = 3600$

where, $^5C_3$ is the number of ways without W_1 and W_2 and $^5C_2$ is the number of ways with W_1 or W_2 but not both.
$\therefore$ Number of ways $= 4200 + 3600 = 7800$

7 Given, X has 7 friends, 4 of them are ladies and 3 are men while Y has 7 friends, 3 of them are ladies and 4 are men.
$\therefore$ Total number of required ways

$= {}^3C_3 \times {}^4C_0 \times {}^4C_0 \times {}^3C_3$
$+ {}^3C_2 \times {}^4C_1 \times {}^4C_1 \times {}^3C_2$
$+ {}^3C_1 \times {}^4C_2 \times {}^4C_2 \times {}^3C_1$
$+ {}^3C_0 \times {}^4C_3 \times {}^4C_3 \times {}^3C_0$
$= 1 + 144 + 324 + 16 = 485$

8 The number of ways of selecting four numbers from 1 to 30 without any restriction is $^{30}C_4$. The number of ways of selecting four consecutive numbers [i.e. (1, 2, 3, 4), (2, 3, 4, 5), ..., (27, 28, 29, 30)] is 27.
Hence, the number of ways of selecting four integers which excludes selection of consecutive four numbers is

$^{30}C_4 - 27 = \dfrac{30 \times 29 \times 28 \times 27}{24} - 27$
$= 27378$

9 Possible even digits are 2,4,6,8,0.
***Case* I** Number has digits 4,6,8,0.
(Here, sum of digits is 18, divisible by 3)

$\square\ \square\ \square\ \square$

Number of arrangements $= 3 \times 3!$
[Ist place can be filled using 4, 6, 8]
$= 3 \times 6 = 18$

***Case* II** Number has digits 2, 4, 6, 0
(Here, sum of digits is 12, divisible by 3)

$\square\ \square\ \square\ \square$

1st place cannot be filled by 0.
Number of arrangements $= 3 \times 3! = 18$
$\therefore$ Number of numbers $= 18 + 18 = 36$

10 $24 = 2 \cdot 3 \cdot 4, 2 \cdot 2 \cdot 6, 1 \cdot 6 \cdot 4, 1 \cdot 3 \cdot 8, 1 \cdot 2 \cdot 12, 1 \cdot 1 \cdot 24$
[as product of three positive integers]
$\therefore$ The total number of positive integral solutions of $xyz = 24$ is

equal to $3! + \dfrac{3!}{2!} + 3! + 3! + 3! + \dfrac{3!}{2!}$ i.e. 30.

Any two of the numbers in each factorisation may be negative. So, the number of ways to associate negative sign in each case is $^3C_2$ i.e. 3.
$\therefore$ Total number of integral solutions
$= 30 + 3 \times 30 = 120$

11 The number of ways to choose zero or more from white balls $= (10 + 1)$
[$\because$ all white balls are mutually identical]
Number of ways to choose zero or more from green balls $= (9 + 1)$
[$\because$ all green balls are mutually identical]
Number of ways to choose zero or more from black balls $= (7 + 1)$
[$\because$ all black balls are mutually identical]
Hence, number of ways to choose zero or more balls of any colour

$= (10 + 1)(9 + 1)(7 + 1)$

Also, number of ways to choose zero balls from the total $= 1$
Hence, the number of ways to choose atleast one ball
[irrespective of any colour]

$= (10 + 1)(9 + 1)(7 + 1) - 1$
$= 880 - 1 = 879$

12 Let $x_1, x_2, \ldots, x_8$ denote the marks assign to 8 questions.
$\therefore x_1 + x_2 + \ldots + x_8 = 30$
Also, $x_1, x_2, \ldots, x_8 \geq 2$

Let, $u_1 = x_1 - 2, u_2 = x_2 - 2 \ldots u_8 = x_8 - 2$
Then, $(u_1 + 2 + u_2 + 2 + \ldots + u_8 + 2) = 30$
$\Rightarrow u_1 + u_2 + \ldots + u_8 = 14$
$\therefore$ Total number of solutions

$= {}^{14+8-1}C_{8-1} = {}^{21}C_7$

13 $2^{10}5^{10}11^{11}13^{13}$ has a divisor of the form $2^\alpha \cdot 5^\beta \cdot 11^\gamma \cdot 13^\delta$, where

$\alpha = 0, 1, 2, \ldots, 10; \beta = 0, 1, 2, \ldots, 10;$
$\gamma = 0, 1, 2, \ldots, 11; \delta = 0, 1, 2, \ldots, 13$

It is of the form $4n + 1$, if

$\alpha = 0; \beta = 0, 1, 2, \ldots, 10;$
$\gamma = 0, 2, 4, \ldots, 10;$
$\delta = 0, 1, 2, \ldots, 13.$

$\therefore$ Number of divisors

$= 11 \times 6 \times 14 = 924$

14 Let the particular side on which 3 particular sailors can work be named A and on the other side by B on which 2 particular sailors can work. Thus, we are left with 3 sailors only. Selection of one sailor for side $A = {}^3C_1 = 3$ and, then we are left with 2 sailors for the other side. Now, on each side, 4 sailors can be arranged in 4! ways.
$\therefore$ Total number of arrangements
$= 3 \times 24 \times 24 = 1728$

15 Required number is the coefficient of x^{10} in $(1 + x + x^2 + \ldots + x^6)^3$

$= (1 - x^7)^3 (1 - x)^{-3} = (1 - 3x^7 + \ldots)$
$\left[1 + \binom{3}{1}x + \binom{4}{2}x^2 + \ldots\right]$

Hence, coefficient of x^{10} is

$\binom{12}{10} - 3\binom{5}{3} = 36.$